Relativistic covariance and rotational electrodynamics

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The following article demonstrates how the logical coherence of relativistic electrodynamics is maintained for a particular family of rotational paradoxes. The internal computational unity, for rotation, is preserved through the manifestation of a commonly unrecognized geometrical property of tensor calculus.

INTRODUCTION

An interesting family of paradoxes frequently discussed in electromagnetics classes concerns the fields produced by rotating charge distributions.[1] That a rotating spherical shell of charge, for example, produces a magnetic field in the frame of a laboratory observer is readily accepted by many students. "However", a student will query, "with respect to an observer whose system of reference is co-rotating with the sphere, the charges are at rest and hence, in this system, no magnetic fields ought to exist."

A similar paradox occurs with rotating cylindrical distributions of charge.[2] Once again, a laboratory observer perceives an axial magnetic field whose source is the rotating cylinder of charge. For a co-rotating observer, the charges are at rest and therefore should produce no magnetic fields. Even worse, for the rotating observer inside the cylinder, by Gauss's Law, there should exist no electric fields. How then can we have a nonzero field tensor inside, in the inertial system (laboratory frame), and a vanishing field tensor in the rotating system, since if a tensor vanishes in any frame it must vanish in all other systems of reference at that point?

In both of these paradoxical examples the reader is cautioned not to accept the conclusions unquestioningly. The logic may be impeccable—but the presuppositions are erroneous. Why do we resurrect these historical paradoxes? Because we believe that they illustrate the computational beauty and conceptual richness of relativity theory as manifested through the inherent presence of the object of anholonomity.

We parenthetically comment that our physics is transpiring on an underlying manifold that has zero curvature (in the limit—i.e., we assume that we can paste charges onto a flat manifold and not disturb the geometrical structure of the manifold). Even though we are doing non-inertial physics, we are properly within the realm of what is *traditionally* called special relativity. (Our approach works equally well on curved spacetimes, of course.)

1. THE ROTATING SHELL OF CHARGE

The presentation of the magnetic field arising from a rotating charged spherical shell appears in many textbooks on electromagnetism. The computation is performed in the frame of an inertial observer and, with the aid of calculational conveniences, smoothly proceeds from a specification of the current density

$$J_\phi(r') = \frac{Q\omega}{4\pi a}\sin\theta'\,\delta(r'-a) \tag{1}$$

to the computation of the vector potential

$$A_\phi(r,\theta) = \begin{cases} \dfrac{Q\omega r}{3ca}\sin\theta & (r\leqslant a) \quad (2a)\\[2ex] \dfrac{Q\omega a^2}{3cr^2}\sin\theta & (r\geqslant a) \quad (2b)\end{cases}.$$

The magnetic induction follows from the curl of the vector potential as

$$B_\theta(r,\theta) = \begin{cases} -\dfrac{2Q\omega}{3ca}\sin\theta & (r<a) \quad (3a)\\[2ex] \dfrac{Q\omega a^2}{3cr^3}\sin\theta & (r>a) \quad (3b)\end{cases}.$$

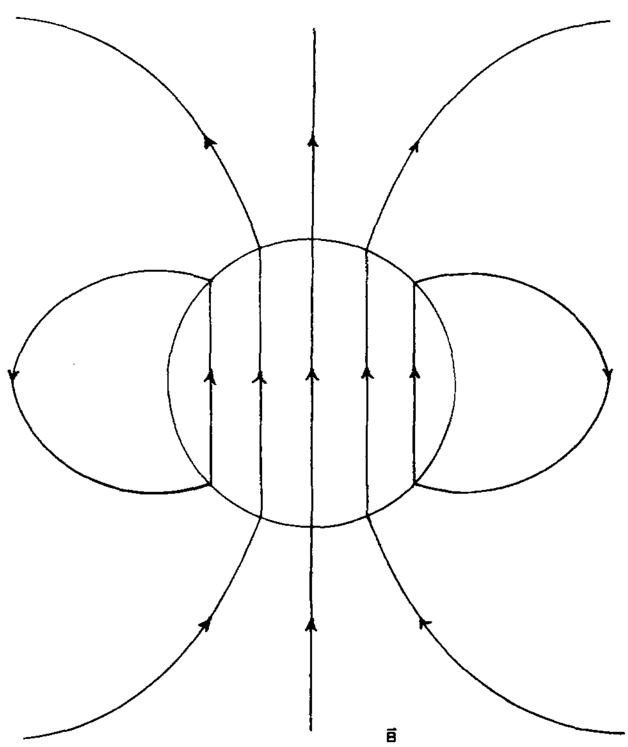

FIG. 1. Magnetic flux density arising from a rotating spherical shell which is uniformly charged as perceived by a nonrotating observer. (Compare figure 5-11a of Ref. 7.)

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and

$$B_r(r,\theta) = \begin{cases} \dfrac{2\omega Q}{3ca}\cos\theta \quad (r<a) & \text{(3c)} \\ \dfrac{2\omega Q a^2}{3cr^3}\cos\theta \quad (r>a) & \text{(3d)} \end{cases}$$

These familiar fields are shown for reference in Fig. 1. The prescription followed has been to specify the current distribution and follow the path $J^{\mu} \rightarrow A^{\mu} \rightarrow F^{\mu\nu}$. The inertial electromagnetic field tensor is taken as

$$F_{\mu\nu} = \frac{\partial A_\nu}{\partial x^\mu} - \frac{\partial A_\mu}{\partial x^\nu}.$$

One wonders if a similar path could be followed in the rest frame of the rotating charges: Say, $J^a \rightarrow A^a \rightarrow F^{ab}$, where the rotating quantities and inertial quantities, by their tensorial nature, would be related by some Lorentz-like transformation. In this article, we propose to pedagogically demonstrate that this covariant nature of Maxwellian electrodynamics, under relativistic rotation, is only attainable with the inclusion of the object of anholonomity. (This remarkable nontensorial object does not modify the theory of relativity in anyway, but rather is a commonly unrecognized[3] inherent pre-supposition of tensor calculus on manifolds[4].)

2. ROTATION IN SPHERICAL COORDINATES

Pirani[5] and later Irvine[6] have discussed how a rotating observer may let his world line provide a time-like direction and employ the Frenet–Serret Formulas to obtain a *field* of orthogonal reference frames, $\mathbf{e}_a$.[4] The result for spherical coordinates, is the *field* of frames ($x^4 = \tau = ct$)

$$\mathbf{e}_1 = \mathbf{e}_r \tag{4a}$$

$$\mathbf{e}_2 = \mathbf{e}_\theta \tag{4b}$$

$$\mathbf{e}_3 = \gamma \mathbf{e}_\phi + \gamma \frac{r^2\omega \sin^2\theta}{c}\mathbf{e}_\tau \tag{4c}$$

$$\mathbf{e}_4 = \gamma \frac{\omega}{c}\mathbf{e}_\phi + \gamma \mathbf{e}_\tau . \tag{4d}$$

These are orthogonal, and are related to the natural basis vectors of the inertial observer's field of reference frames, at every observable spacetime event, by the Lorentz-like transformation

$$\mathbf{e}_a = h^{\mu}_{\ a}\mathbf{e}_\mu , \tag{5}$$

where

$$h^{\mu}_{\ a} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \gamma & \gamma \dfrac{\omega}{c} \\ 0 & 0 & \gamma \dfrac{r^2\omega\sin^2\theta}{c} & \gamma \end{pmatrix} \tag{6}$$

and

$$\gamma = \left(1 - \frac{r^2\omega^2\sin^2\theta}{c^2}\right)^{-\frac{1}{2}}. \tag{7}$$

Furthermore, for all observers in the rotating frame, the metric tensor $g_{ab} = \mathbf{e}_a \cdot \mathbf{e}_b$, will then have the orthogonal form

$$g_{ab} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & r^2 & 0 & 0 \\ 0 & 0 & r^2\sin^2\theta & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}. \tag{8}$$

Notice that this is consistent with $g_{ab} = h^{\mu}_{\ a} h^{\nu}_{\ b} g_{\mu\nu}$, where $g_{\mu\nu}$ is the metric of the inertial (or non-rotating) frame.

In the inertial frame, one employs the usual spherical coordinate Christoffel Symbols

$$\begin{Bmatrix} \mu \\ \nu\alpha \end{Bmatrix} = \tfrac{1}{2} g^{\mu\beta}\left[g_{\alpha\beta,\nu} + g_{\beta\nu,\alpha} - g_{\nu\alpha,\beta}\right]. \tag{9}$$

These familiar non-tensorial objects are symmetric in their lower indices, and their spherical coordinate values appear in many textbooks. Given the components with respect to one frame, one transforms them with respect to any other frame according to

$$\Gamma^{a}_{\ bc} = h^{a}_{\ \mu} h^{\alpha}_{\ b} h^{\nu}_{\ c} \begin{Bmatrix} \mu \\ \nu\alpha \end{Bmatrix} - h^{\mu}_{\ b} h^{\nu}_{\ c} \frac{\partial h^{a}_{\ \nu}}{\partial x^\mu}. \tag{10}$$

Employing Eq. 6, one finds that even though the inertial frame $\begin{Bmatrix} \mu \\ \nu\alpha \end{Bmatrix}$ is symmetric in the lower indices, the $\Gamma^{a}_{\ bc}$ are not. This is because the set of frames, Eq.(4) and the transformation, Eq.(6) are *anholonomic*. The concept of anholonomity has been discussed elsewhere and one should note that it arises from a choice of the field of reference frames and is not a tensorial quantity on the underlying manifold as, for example, torsion would be. (Torsion can't be transformed away over an extended region.) How does this asymmetry affect electrodynamics?

An invariant form of Maxwell's Equations may be arrived at from a variational principle as

$$\nabla_\nu F^{\mu\nu} = \frac{4\pi}{c} J^{\mu}, \tag{11}$$

where

$$J^{\mu} = \rho v^{\mu} \tag{12}$$

and

$$F_{\mu\nu} = \nabla_\mu A_\nu - \nabla_\nu A_\mu . \tag{13}$$

The four-vector potential has the covariant components given by:

$$A_\mu = (A_r, rA_\theta, r\sin\theta A_\phi; -\Phi). \tag{14}$$

In inertial frames of reference, Eq. (13) reduces to the simple expression

$$F_{\mu\nu} = \frac{\partial A_\nu}{\partial x^\mu} - \frac{\partial A_\mu}{\partial x^\nu}. \tag{15}$$

However, in the rotating frames of Eq. (4), Eq. (13) must be written as the tensor

$$F_{ab} = \frac{\partial A_b}{\partial x^a} - \frac{\partial A_a}{\partial x^b} + 2\,\Omega^{\ c}_{ab} A_c , \tag{16}$$

where

$$\Omega^{\ c}_{ba} \triangleq \tfrac{1}{2}\left[\Gamma^{c}_{\ ab} - \Gamma^{c}_{\ ba}\right]. \tag{17}$$

In inertial frames, this last component vanishes, but in rotating frames, Eq. (15) is inappropriate for the description of electrodynamics because it leaves out anholonomic effects (Eq. (15) is *not* a tensor unless $\Omega^{\ \ \alpha}_{\nu\mu} = 0$). The objects of anholonomity may be computed by a variety of techniques,[4] and since they are needed here, we present their computation by the Cartan Calculus in the Appendix.

3. SPECIFICATION OF THE SOURCE DISTRIBUTION

In the inertial frame, let us specify the components of the four–current density

$$J^{\mu} = \rho v^{\mu} = \left(J_r, \frac{J_\theta}{r}, \frac{J_\phi}{r\sin\theta}; c\rho\right) \tag{18}$$

so that

$$J^{\mu} = (0,0,\omega\rho_0; c\rho_0)\,. \tag{19}$$

The charge density is taken as *uniform when observed from the inertial frame*, say

$$\rho_0 = \frac{Q}{4\pi a}\,\delta(r' - a)\,.$$

In particular, for computational convenience, this means that we are assuming that *in the proper frame of the shell* the density of charge varies continuously with latitude in such a way that the charge distribution just compensates for the relativistic increase in density and hence, in the inertial frame, is perceived as a uniform charge distribution. [This assumption was really made back in Eq. (1).] In the inertial frame, the differential equation represented by 11, 12 and 13 may be solved as

$$A_4 = -\Phi = \begin{cases} -\dfrac{Q}{a} & r \leqslant a \\ -\dfrac{Q}{r} & r \geqslant a \end{cases} \tag{20}$$

and

$$A_3 = A_\phi r\sin\theta = \begin{cases} \dfrac{Q\omega r^2}{3ca}\sin^2\theta & r \leqslant a \\ \dfrac{Q\omega a^2}{3rc}\sin^2\theta & r \geqslant a\,. \end{cases} \tag{21}$$

One may quickly form the inertial frame field tensor from Eq. (15) (since $\Omega^{\ \ \alpha}_{\nu\mu} = 0$). Further, the separate $F^{\mu\nu}$ for $r < a$ and $r > a$ satisfy the point-wise boundary conditions across the shell discontinuity. We have followed the prescription given in Sec. 1: $J^{\mu} \rightarrow A^{\mu} \rightarrow F^{\mu\nu}$. We now shift to the rotating frame and pursue the suggestion $J^a \rightarrow A^a \rightarrow F^{ab}$. If our analysis is acceptable, we should have a completely covariant formulation of the problem, and have resolved any paradoxes along the way.

4. COMPUTATIONS IN THE ANHOLONOMIC FRAME

Employing the dual to the transformation of Eq. (6), we write down the current density in the proper frame of the charges (the rotating frame)

$$J^a = h^{\ a}_{\mu} J^{\mu} = \rho v^a \tag{22}$$

or, more explicitly

$$J^a = \left(0,0,0; c\frac{\rho_0}{\gamma}\right). \tag{23}$$

Again, J^4 reflects our choice of having ρ_0 specified as uniform in the inertial frame. This form for the charge density is acceptable because the rotating observer perceives no moving charges (and hence no apparent magnetic field-producing currents.)

Remembering that in the rotating frame $dV = \gamma dV_0$, we note that charge invariance is satisfied:

$$\frac{1}{c}\iiint J^a dS_a = Q\,. \tag{24}$$

Since Eq(11) is form-invariant, the vector potential must satisfy the differential equation

$$\nabla_b\left[g^{ac}g^{bd}\left(\frac{\partial A_d}{\partial x^c} - \frac{\partial A_c}{\partial c^d} + 2\,\Omega^{\ \ e}_{cd}A_e\right)\right] = \frac{4\pi}{c}J^a, \tag{25}$$

where the objects of anholonomity are given in the Appendix. One may find the transformed components of the vector potential by Eq. (6)

$$A_a = h^{\ \mu}_{a} A_\mu$$

as

$$A_1 = A_2 = 0, \tag{26a}$$

$$A_3 = \begin{cases} -\dfrac{2\gamma Q r^2 \omega \sin^2\theta}{3ca} & (r \leqslant a) \\ \dfrac{\gamma Q \omega a^2 \sin^2\theta}{3cr} - \dfrac{\gamma Q r \omega \sin^2\theta}{c} & (r \geqslant a) \end{cases} \tag{26b}$$

$$A_4 = \begin{cases} -\dfrac{\gamma Q}{a} + \dfrac{\gamma Q r^2 \omega^2 \sin^2\theta}{3c^2 a} & (r \leqslant a) \\ -\dfrac{\gamma Q}{r} + \dfrac{\gamma Q a^2 \omega^2 \sin^2\theta}{3rc^2} & (r \geqslant a). \end{cases} \tag{26c}$$

Now, by Eq. (26), (8), and (23) and the tabulated $\Omega^{\ \ a}_{bc}$, one may readily verify that Eq. (26) is indeed the solution of Eq.

$$\Omega^{\ \ c}_{ab} \neq 0$$

$$\begin{tikzcd}
J^{\mu} \arrow[r] \arrow[d, leftrightarrow, "h^{\ a}_{\mu}"] & A^{\mu} \arrow[r] \arrow[d, leftrightarrow, "h^{\ a}_{\mu}"] & F^{\mu\nu} \arrow[r] \arrow[d, leftrightarrow, "h^{\ a}_{\mu} h^{\ b}_{\nu}"] & \nabla_\nu F^{\mu\nu} = \frac{4\pi}{c} J^{\mu} \arrow[d, leftrightarrow, "h^{\ a}_{\mu}"] \\
J^a \arrow[r] & A^a \arrow[r] & F^{ab} \arrow[r] & \nabla_b F^{ab} = \frac{4\pi}{c} J^a
\end{tikzcd}$$

Inertial Frame: $F_{\mu\nu} \doteq \nabla_\mu A_\nu - \nabla_\nu A_\mu = \dfrac{\partial A_\nu}{\partial X^\mu} - \dfrac{\partial A_\mu}{\partial X^\nu} + \overbrace{\left(\Gamma^{\ \alpha}_{\nu\mu} - \Gamma^{\ \alpha}_{\mu\nu}\right)}^{0} A_\alpha$

Rotating Frame: $F_{ab} \doteq \nabla_a A_b - \nabla_b A_a = \dfrac{\partial A_b}{\partial X^a} - \dfrac{\partial A_a}{\partial X^b} + 2\,\Omega^{\ \ c}_{ab} A_c$

FIG. 2. This diagram indicates what is demanded of any covariant formulation of electrodynamics. A relativistically covariant formulation is possible for rotation by virtue of the intrinsic nature of the anholonomic object.

(25). What would have happened if the rotating observer had neglected the anholonomic contribution? Figure 2 would then have to display an internal inconsistency.

Next, one might desire the field tensor in the rotating frame. The reader is encouraged to transform the inertial $F_{\mu\nu}$ and compare with the results computed by Eq. (16) and the tabulated $\Omega^{\;c}_{ab}$. For completeness, we list the nonzero results (as obtained by either method).

$$F_{14} = E_r = \begin{cases} \dfrac{2\gamma Q r \omega^2 \sin^2\theta}{3ac^2} & (r<a) \\ \dfrac{\gamma Q}{r^2}\left(1 - \dfrac{a^2\omega^2\sin^2\theta}{3c^2}\right) & (r>a) \end{cases} \tag{27a}$$

$$F_{24} = rE_\theta = \begin{cases} \dfrac{2\gamma Q r^2 \omega^2 \sin\theta\cos\theta}{3ac^2} & (r<a) \\ \dfrac{2\gamma Q a^2 \omega^2 \sin\theta\cos\theta}{3rc^2} & (r>a) \end{cases} \tag{27b}$$

$$F_{31} = r\sin\theta B_\theta = \begin{cases} -\dfrac{2\gamma Q r \omega \sin^2\theta}{3ac} & (r<a) \\ -\dfrac{\gamma Q \omega \sin^2\theta}{c}\left(1 - \dfrac{a^2}{3r^2}\right) & (r>a) \end{cases} \tag{27c}$$

$$F_{23} = r^2\sin\theta B_r = \begin{cases} \dfrac{2\gamma Q r^2 \omega \sin\theta\cos\theta}{3ac} & (r<a) \\ \dfrac{2\gamma Q a^2 \omega \sin\theta\cos\theta}{3rc} & (r>a). \end{cases} \tag{27d}$$

One may verify that Eqs. 27 really do satisfy the Maxwell Equations (if, by now, he is not already convinced of the built in anholonomity in relativity for non-inertial frames). This exercise is particularly illuminating since the left-hand side of Maxwell's source equations [or Eq. (25)] vanishes for $r \neq a$, and the jump conditions are satisfied across the shell discontinuity.

5. CYLINDRICAL SHELL OF CHARGE

When discussing the rotating cylinder, Fig. 3, Feynman[2] makes the provocative comment, "There is no 'relativity of rotation'. A rotating system is not an inertial frame, and the laws of physics are different. We must be sure to use equations of electromagnetism only with respect to inertial coordinate systems." To which we agree wholeheartedly. But, after making this seductive and tantalizing statement, he passes on to another topic without hinting how one does do electromagnetism in noninertial systems. (At a similar point in their analysis of the sphere, Panofsky and Phillips[1] appeal to General Relativity, even though they are working in a flat spacetime.) Surely one may proceed as $J^a \to A^a \to F^{ab}$ since all are tensor quantities.

Let us formulate the problem in the inertial system in the following manner: we specify the current density in cylindrical coordinates.

$$J^\mu = \rho v^\mu = (J_r, J_\theta/r, J_z; c\rho) \tag{28}$$

as

$$J^\mu = (0, \omega\sigma_0, 0, c\sigma_0), \tag{29}$$

where σ_0 is the surface charge density, assumed to be uniform over the thin cylindrical shell as perceived by the inertial observer. We write

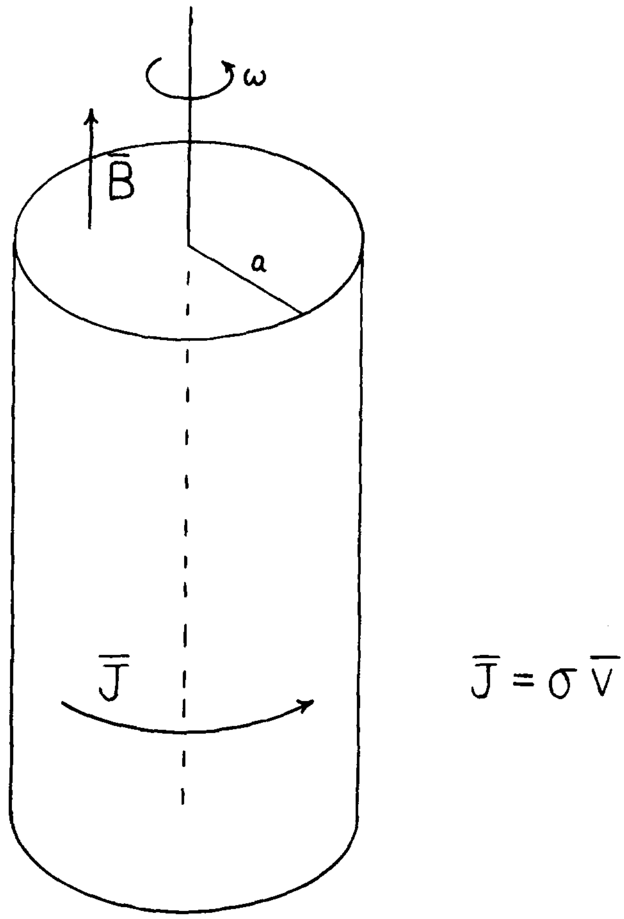

FIG. 3. Magnetic flux density arising from a rotating charged cylindrical shell. (Compare Fig. 14-5 of Ref. 2)

$$\sigma_0 = \frac{Q}{2\pi a L}\,\delta(r' - a) = \frac{\lambda}{2\pi a}\,\delta(r' - a). \tag{30}$$

Equations (11), (12), and (13) are then solved as

$$A_2 = rA_\theta = \begin{cases} \dfrac{2\pi\sigma_0 a\omega r^2}{c} & (r \leqslant a) \\ \dfrac{2\pi\sigma_0 a^3\omega}{c} & (r \geqslant a) \end{cases} \tag{31a}$$

$$A_4 = -\Phi = \begin{cases} \Phi_0 & (r \leqslant a) \\ \Phi_0 + \dfrac{\lambda}{2\pi}\ln\left(\dfrac{a}{r}\right), & (r \geqslant a) \end{cases} \tag{31b}$$

where Φ_0 is a suitably chosen constant. The field tensor has nonzero components:

$$F_{14} = E_r = \begin{cases} 0 & r<a \\ \dfrac{\lambda}{2\pi r} & r>a \end{cases} \tag{32}$$

$$F_{12} = rB_z = \begin{cases} \dfrac{4\pi\sigma_0 a\omega r}{c} & r<a \\ 0 & r>a. \end{cases} \tag{33}$$

We now turn to the analysis $J^a \to A^a \to F^{ab}$ in the rotating frame. In this system

$$J^a = \rho v^a = (0,0,0,c\sigma_0/\gamma). \tag{34}$$

Here again, we have assumed that in the proper frame of the cylinder the density of charge varies in such a way that the charge distribution just compensates for the relativistic increase in density and is consequently perceived *by the inertial observer* as uniform. (In this case a naive application of Gauss's Law would surely lead a rotating observer to conclude that F^{ab} for $r < a$ vanishes entirely.) The $\Omega_{ab}^{\ \ c}$ for rotating cylindrical coordinates have been included in the Appendix. One may readily verify that a solution to Eq.(25) for the distribution of Eq. (34) is

$$A_2 = \frac{2\pi\gamma\sigma a\omega r^2}{c} + \frac{\gamma\omega\Phi_0 r^2}{c} \quad (r \leqslant a) \tag{35a}$$

$$A_4 = \gamma\Phi_0 + \frac{2\pi\gamma\sigma_0 a\omega^2 r^2}{c^2} \quad (r \leqslant a). \tag{35b}$$

From Eq. (16) we compute the nonzero components of the field tensor as

$$F_{12} = E_r = \frac{4\pi\sigma_0 a\omega^2 r}{c^2} \quad (r < a) \tag{36a}$$

$$F_{12} = rB_z = \frac{4\pi\gamma\sigma_0 r a\omega}{c} \quad (r < a). \tag{36b}$$

We note that these satisfy the Maxwell Equations. As a check, we also see that

$$J^a = h_{\ \mu}^{a} J^\mu \tag{37a}$$

$$A^a = h_{\ \mu}^{a} A^\mu \tag{37b}$$

$$F_{ab} = h_{\ a}^{\mu} h_{\ b}^{\nu} F_{\mu\nu}. \tag{37c}$$

The internal consistancy of Fig. 2 is again demonstrated. We are now in a position to analyze Feynman's students' query, " 'What if I put myself in the frame of reference of the rotating cylinder? Then there is just a charged cylinder at rest, and I know that the electrostatic equations say there will be *no* electrostatic fields inside . . . . Something must be wrong.' "

Our response is to reecho our opening comments: the logic is unquestionable, but the presuppositions (concerning $\Omega_{ab}^{\ \ c}$) are unsound.

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APPENDIX

For reasons of completeness, we briefly sketch one of several techniques for obtaining the spherical coordinate objects of anholonomity for the rotating observer. (The cylindrical coordinate objects are derived in Ref. 4 .) In order to obtain the $\Omega_{ab}^{\ \ c}$, one may actually perform the laborious calculation indicated by Eq. (10) of this article. Alternatively, he might employ the field frames given by Eq.(4) and the duality relation:

$$\langle \boldsymbol{\omega}^a, \mathbf{e}_b \rangle = \delta_b^a \tag{A1}$$

to find the natural 1–forms for the rotating observer.

$$\boldsymbol{\omega}^1 = dr \tag{A2a}$$

$$\boldsymbol{\omega}^2 = d\theta \tag{A2b}$$

$$\boldsymbol{\omega}^3 = \gamma(d\phi - \omega dt) \tag{A2c}$$

$$\boldsymbol{\omega}^4 = \gamma c\left(dt - \frac{r^2\omega \sin^2\theta}{c^2} d\phi\right). \tag{A2d}$$

Given these forms, one may compute the exterior derivative

$$d\boldsymbol{\omega}^a = 2\Omega_{bc}^{\ \ a} \boldsymbol{\omega}^b \wedge \boldsymbol{\omega}^c \tag{A3}$$

and read off the non-zero components of the spherical objects of anholonomity:

$$\Omega_{13}^{\ \ 3} = -\Omega_{31}^{\ \ 3} = \tfrac{1}{2} \frac{\gamma^2 r\omega^2 \sin^2\theta}{c^2} \tag{A4a}$$

$$\Omega_{23}^{\ \ 3} = -\Omega_{32}^{\ \ 3} = \tfrac{1}{2} \frac{\gamma^2 r^2\omega^2 \sin\theta\cos\theta}{c^2} \tag{A4b}$$

$$\Omega_{41}^{\ \ 4} = -\Omega_{14}^{\ \ 4} = \tfrac{1}{2} \frac{\gamma^2 r\omega^2 \sin^2\theta}{c^2} \tag{A4c}$$

$$\Omega_{42}^{\ \ 4} = -\Omega_{24}^{\ \ 4} = \tfrac{1}{2} \frac{\gamma^2 r^2\omega^2 \sin\theta\cos\theta}{c^2} \tag{A4d}$$

$$\Omega_{31}^{\ \ 4} = -\Omega_{13}^{\ \ 4} = \gamma^2 \frac{r\omega \sin^2\theta}{c} \tag{A4e}$$

$$\Omega_{32}^{\ \ 4} = -\Omega_{23}^{\ \ 4} = \gamma^2 \frac{r^2\omega \sin\theta\cos\theta}{c} \tag{A4f}$$

Similarly, the nonzero cylindrical coordinate components of the anholonomic objects are:

$$\Omega_{12}^{\ \ 2} = -\Omega_{21}^{\ \ 2} = +\frac{\gamma^2 r\omega^2}{2c^2} \tag{A5a}$$

$$\Omega_{21}^{\ \ 4} = -\Omega_{12}^{\ \ 4} = +\frac{\gamma^2 r\omega}{c} \tag{A5b}$$

$$\Omega_{41}^{\ \ 4} = -\Omega_{14}^{\ \ 4} = +\frac{\gamma^2 r\omega^2}{2c^2} \tag{A5c}$$

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